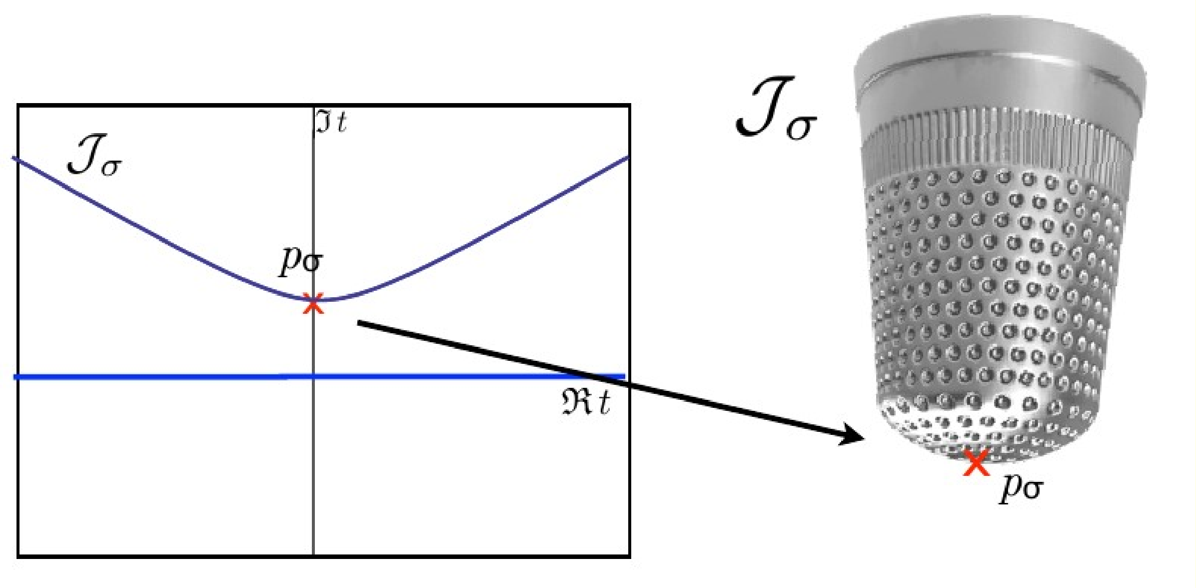
Solution of simple toy models via thimble regularization of lattice field theory

Giovanni Eruzzi
(and F. Di Renzo)

University of Parma and INFN



The Sign Problem

$$\langle O \rangle = N \int \mathcal{D}\Phi \, O\left[\Phi\right] e^{-S[\Phi]}$$

- $S \in \mathbb{C}$ → ~~Probability weight for Monte-Carlo~~
- Oscillatory part $S^I = \Im\left(S\right)$ scales exponentially with V!

May prevent reweighting!

⇒ Let's study an alternative approach!

Picard-Lefschetz (complex Morse) theory: a primer for 0-dimensional toy-models

$$\langle O \rangle \sim \int_{\mathbb{R}} dx\, O(x)\, e^{-S(x)} = \sum_{\sigma} m_{\sigma} \int_{\mathcal{J}_{\sigma}} dz\, O(z)\, e^{-S(z)}$$

- $m_\sigma \in \mathbb{Z} \longrightarrow$ integer coefficients
- $\{\mathcal{J}_\sigma\} \longrightarrow$ Lefschetz thimbles
- $S^I = \Im(S) \longrightarrow$ constant along $\mathcal{J}_\sigma$!

[E. Witten, Analytic continuation of Chern-Simons Theory, arXiv:1001.2933v4 [hep-th]]

As S^I is constant along $\mathcal{J}_\sigma$, $\exp\left(-S\left(z\right)\right)|_{\mathcal{J}_\sigma}$
can be used as a probability weight

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OK for Monte-Carlo!

But..

- How is a "thimble" defined?
- How are the coefficients $\{m_\sigma\}$ computed?
- How do we integrate over $\mathcal{J}_\sigma$?

Let's pause for a moment...

Why 0-dimensional QFT?

- Useful as toy-models
- Often problematic for complex Langevin
- Often non trivial for Morse theory

→ Worth studying!

Let z_σ label a (complex) critical point of the (complexified) action:

→ Steepest-Ascent equations for $S^R = \Re\left(S\right)$

$$\begin{cases} \dot{x} = +\frac{\partial S^R}{\partial x} \\ \dot{y} = +\frac{\partial S^R}{\partial y} \end{cases}$$

with the boundary condition: $\lim_{t \to -\infty} z\left(t\right) = z_\sigma$

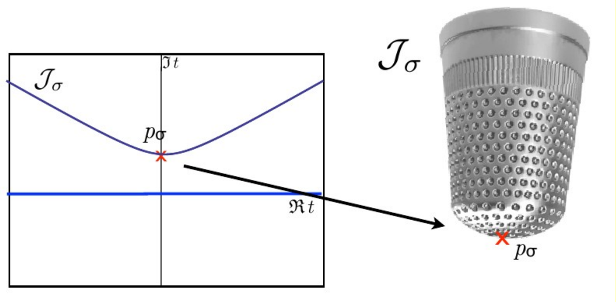
$z = x + i\,y$ → complexified "field"

$z\left(t\right)$ is a curve in the complex plane (Lefschetz thimble associated with the critical point z_σ)

Hessian of S^R at critical point z_σ

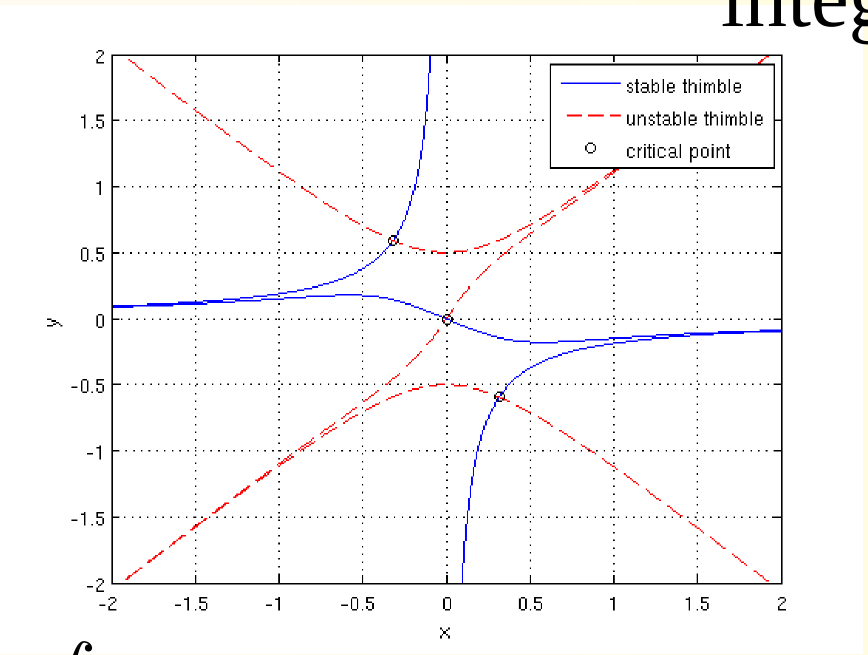
$$H(x_\sigma, y_\sigma) = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

(by holomorphicity)



- Two opposite eigenvalues (λ^+, λ^-)
- Thimble tangent space at critical point = <u>eigenvector</u> with positive eigenvalue: $|v^+\rangle$

Start near z_σ, along $\left|v^+\right\rangle$; now integrate steepest-ascent equations



$$z\left(t\right)$$

Lefschetz thimble

$$\int_{\mathcal{J}_\sigma} dz\, O\left(z\right) e^{-S(z)} = e^{-i\, S^I(z_\sigma)} \int dt\, O\left(z\left(t\right)\right) e^{-S^R(z(t))} z'\left(t\right)$$

Note: 1 dimensional thimble $\longleftrightarrow$ two curves from one critical point (initial condition on $\pm\left|v^+\right\rangle$)

- S^I is constant along $\mathcal{J}_\sigma$ → can be factored out of the integral → same value as at critical point!

- $z'(t) = |z'(t)|\, e^{i\,\phi(t)}$; $\phi(t)$ → "residual phase"

potential source of a "residual sign problem" (of course this can be trivially computed for 0-dimensional toy-models; in general we expect it to be rather smooth)

$|\phi(t)| << 1$ for a real-life model...

[Kikukawa et al, Hybrid Monte Carlo on Lefschetz thimbles - A study of the residual sign problem, JHEP 1310 (2013), 147]

[Cristoforetti et al, An efficient method to compute the residual phase on a Lefschetz thimble, arXiv:1403.5637 [hep-lat], to appear on Phys Rev D]

$m_\sigma \longrightarrow$ Number of intersections between the unstable thimble $\mathcal{K}_\sigma$ and the original domain of integration $\mathbb{R}$

$\mathcal{K}_\sigma$: solution of steepest-descent equations $(\dot{x}, \dot{y}) = -\nabla S^R$

with tangent space at critical point = eigenvector with negative eigenvalue: $|v^-\rangle$

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of $\partial^2 S^R(z_\sigma)$

In principle tricky for multi-dimensional integrals...

The 0-dimensional ϕ^4 theory

$$S(\phi) = \frac{1}{2}\sigma\,\phi^2 + \frac{1}{4}\lambda\,\phi^4$$

- $\sigma = \sigma_R + i\,\sigma_I \in \mathbb{C}$
- $\lambda \in \mathbb{R}^+$

[J. Ambjørn, S. K. Yang, Numerical problems in applying the Langevin equation to complex effective actions, Phys. Lett. B 165 (1985) 140-146]

- Introduced in 1985 (aim of the study was complex Langevin)
- Analytical results readily available

For current status of pros and cons of the complex Langevin approach, refer to

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G. Aarts, P. Giudice, E. Seiler, Localised distributions and criteria for correctness in complex Langevin dynamics, Annals Phys. Vol. 337 (2013) 238-260

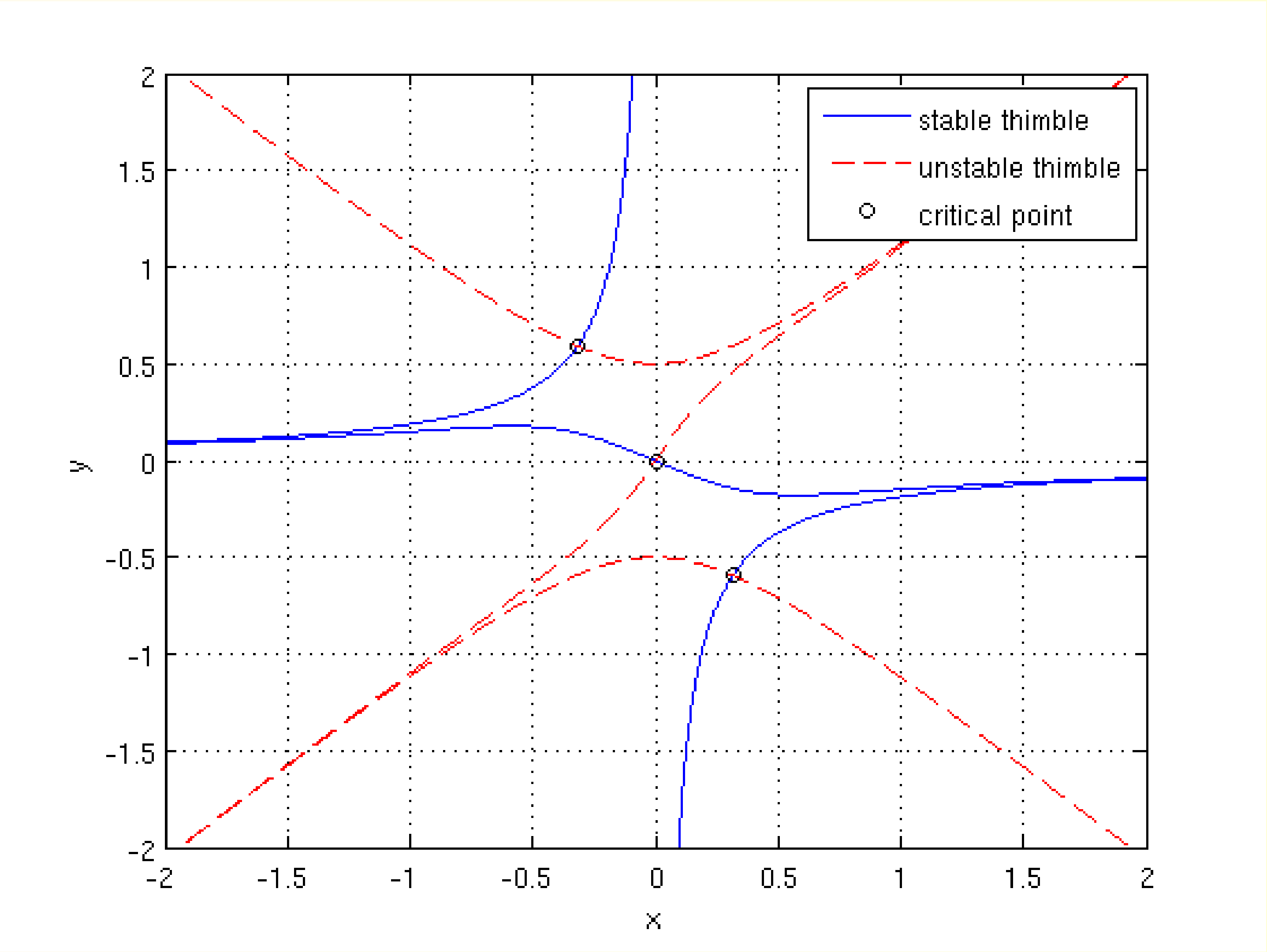
- Divergence of high order momenta: $\langle \phi^n \rangle$ for $n > 4$ in certain regions of parameters $\longrightarrow$ σ_R vs σ_I
- Very unstable complex Langevin dynamics for $\sigma_R < 0$
- On the other side, highly non trivial thimble structure

Three critical points

$$z_0 = 0 \qquad\qquad z_\pm = \pm\sqrt{-\sigma/\lambda}$$

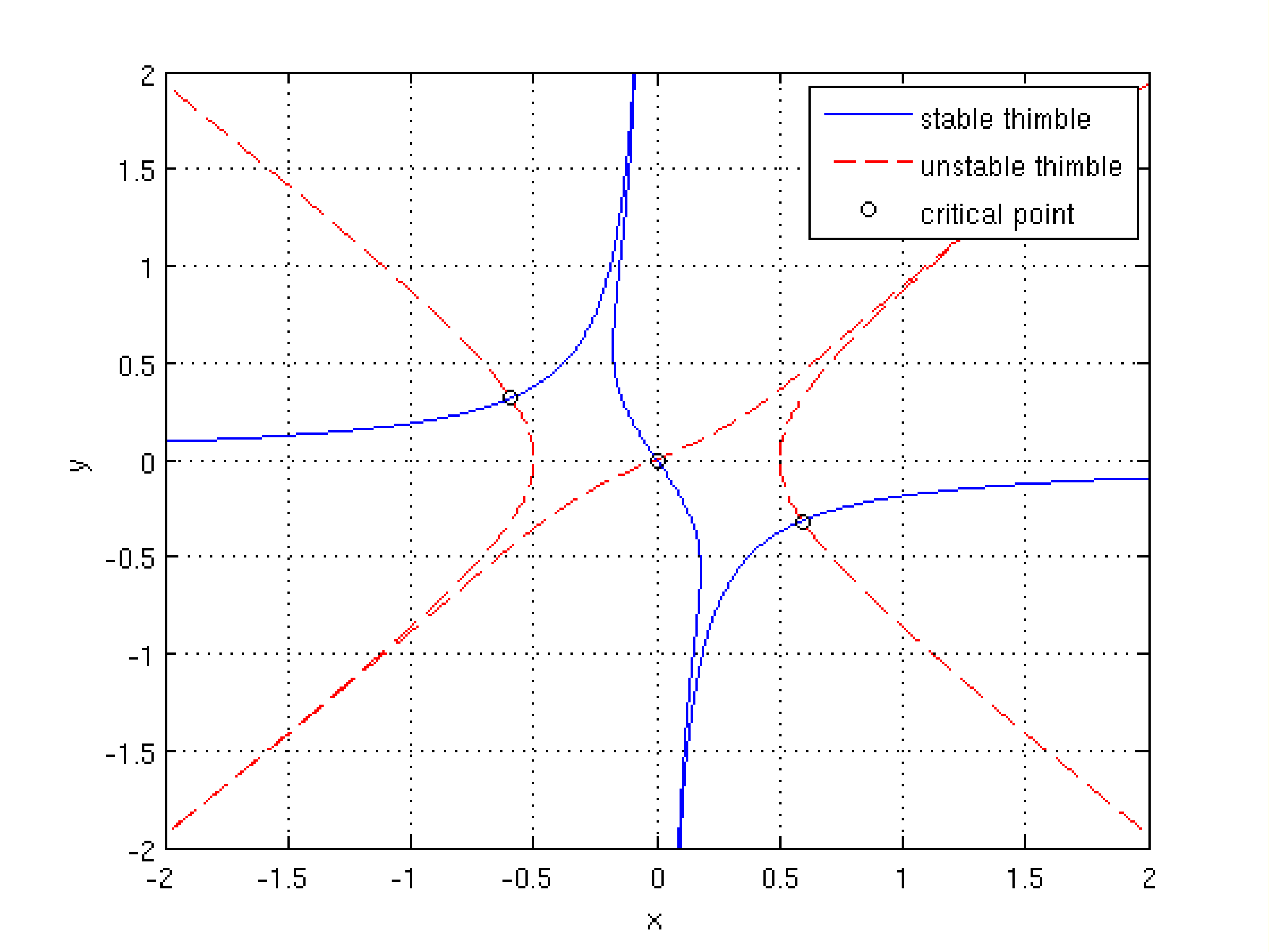
- Different $\{m_\sigma\}$ for opposite signs of σ_R
- Stokes phenomenon for $\sigma_R = 0$

$$\sigma = +0.5 + 0.75\,i \qquad \lambda = 2$$



$$\sigma_R > 0 \longrightarrow m_0 = +1 \quad m_+ = m_- = 0$$

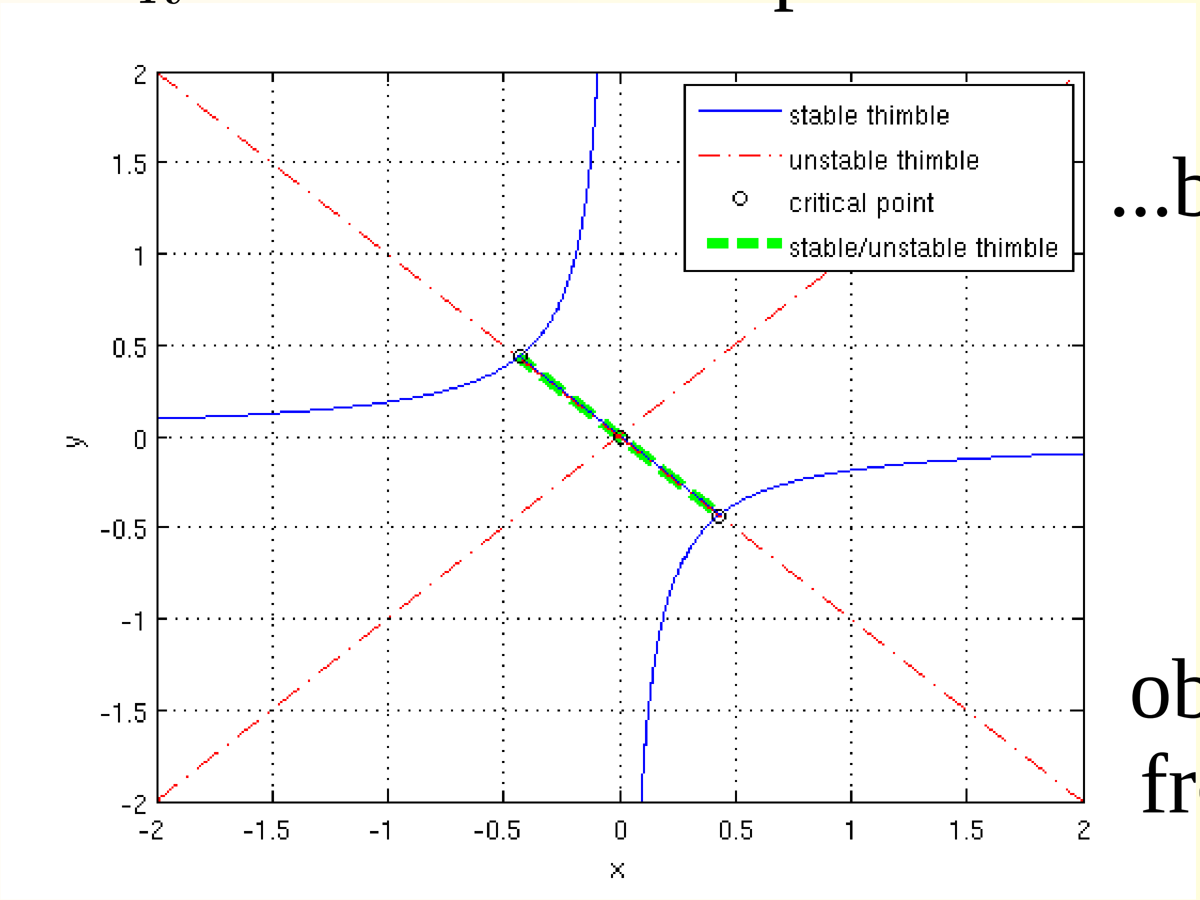
$$\sigma = -0.5 + 0.75\,i \qquad \lambda = 2$$



$$\sigma_R < 0 \longrightarrow m_0 = -1 \quad m_+ = m_- = +1$$

When σ crosses the line $\sigma_R = 0$ in the complex σ plane:

$\sigma_R = 0 \longrightarrow$ Stokes phenomenon $\longrightarrow$ jump in $\{m_\sigma\}$



...but $S(\phi)$ is holomorphic in $\sigma \in \mathbb{C}$!

there must be no discontinuity in the observables while going from $\sigma_R = 0^+$ to $\sigma_R = 0^-$

By imposing continuity, we get the right sign combination!

- Compute $\langle \phi^n \rangle$ for any given value of (σ, λ)

- Perform numerical integration on the thimbles as described above

- Employ different types of Monte-Carlo!

- Metropolis-like thimble Monte-Carlo

[A. Mukherjee, M. Cristoforetti, L. Scorzato, Metropolis Monte~Carlo integration on the Lefschetz thimble: Application to a one-plaquette model, Phys Rev D 88, 051502 (2013)]

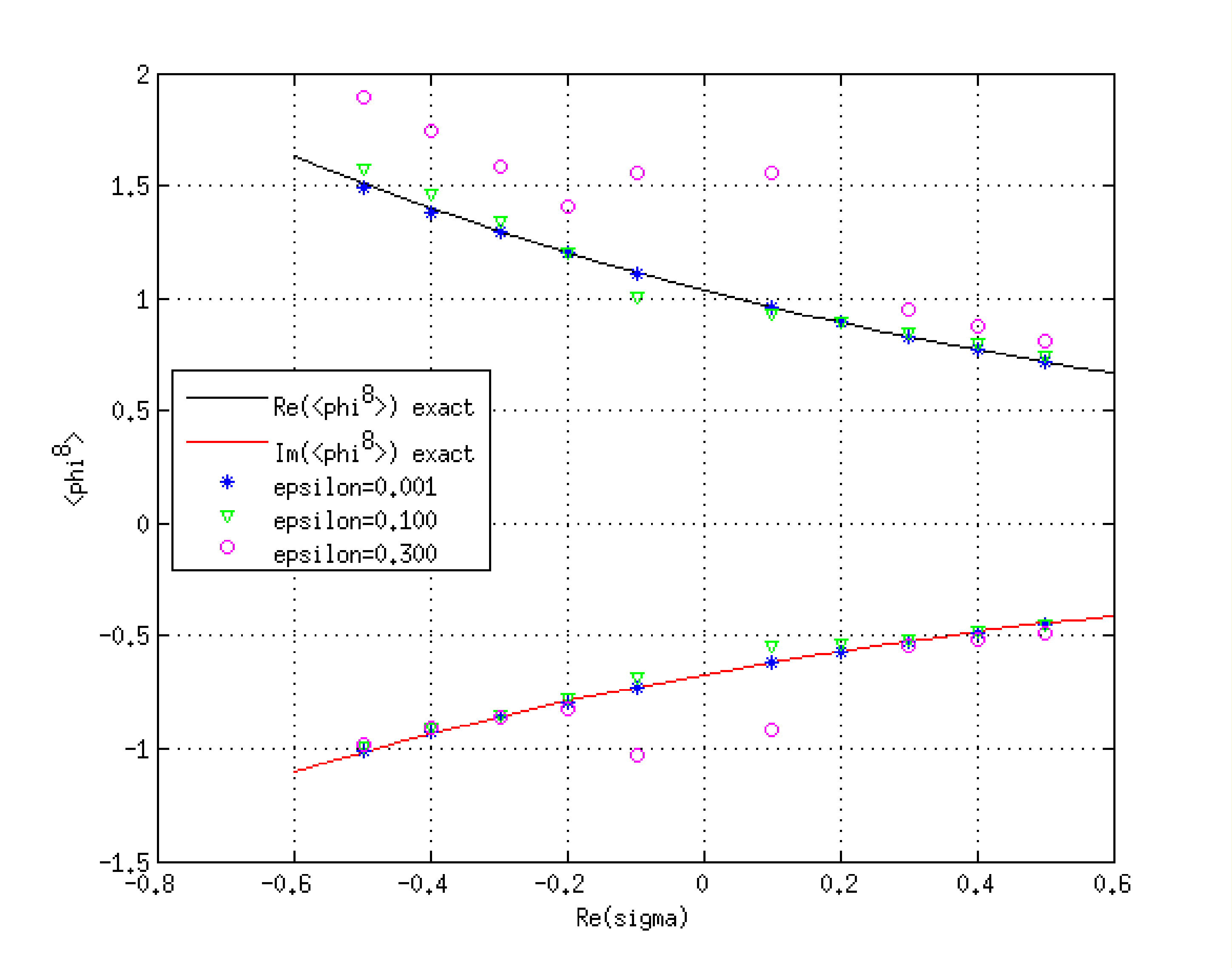
- Langevin-like thimble Monte-Carlo (Aurora algorithm)

Trick: tangent space to the thimble is locally given by ∇S^R

[M. Cristoforetti, F. Di Renzo, L. Scorzato, New approach to the sign problem in quantum field theories: High density QCD on a Lefschetz thimble, Phys. Rev. D 86, 074506 (2012)]

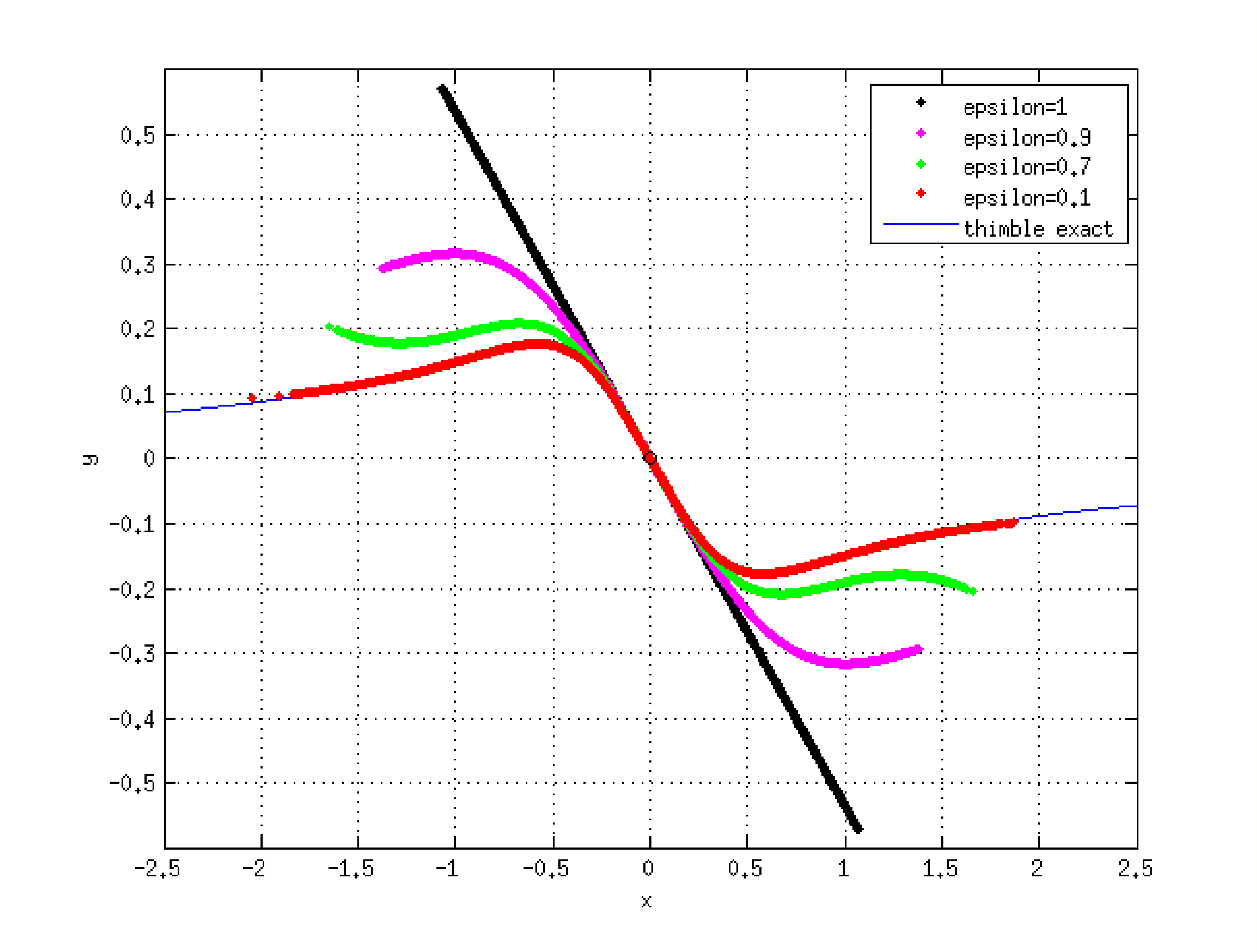
- "Ideal sampling" along the thimble

F. Di Renzo's talk on friday...



ε → technical parameter of the simulation: the smaller, the better the thimble is covered

Monte-Carlo algorithms succeed in covering the thimble!



A brief summary:

- Formulation of a QFT on a Lefschetz thimble is a promising way to solve the 'sign problem'.
- Zero-dimensional toy-models can be extremely useful to understand this type of formulation.
- Different Monte-Carlo algorithms can be devised to do importance sampling over Lefschetz thimbles.

Thanks for your attention!!!